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# Note

## Vortex Tangle Simulations

As a result of the measurement of the attenuation of second sound in rotating superfluid $^4$He [1, 2] the turbulent state of superfluid $^4$He has been modeled for decades [3–5] as a self-sustaining tangle of quantized vortex lines. Paradoxically, even though quantized vortices behave like classical ideal vortices Schwarz [6–10] claims to be able to accurately model a vortex tangle with the self-induction approximation [11], which is known to be a poor approximation to Euler's equation. Our work [12–14] has resolved this paradox by showing that if the numerical calculations are underresolved there is an artificial generation of vortices at small scales in the self-induction model equation which is not present when the equation is solved correctly.

For completeness we briefly describe the equation that is being solved and the reconnection algorithm. The goal of the numerical calculations is to calculate the equilibrium line length density $L$ of vortices present in a vortex tangle; this was attempted by writing the evolution equation for an individual vortex,

$$\frac{\partial \mathbf{r}}{\partial t} = \beta \mathbf{r}' \times \mathbf{r}'' + \alpha \mathbf{r}' \times \mathbf{w}_0 + \beta \alpha \mathbf{r}'', \qquad (1)$$

where $\mathbf{r}$ is the position of the vortex, $t$ is time, $'$ indicates derivative with respect to arclength, $\mathbf{w}_0$ is the relative velocity between the normal and superfluid components, $\beta = (\kappa/4\pi)\ln(R/a_0)$, $\kappa$ is the quantum of circulation, $R$ is the radius of curvature, $a_0$ is the core radius ($\approx 1.2 \times 10^{-8}$ cm), $\alpha = (\rho_n B)/(2\rho)$, $B$ is the Hall–Vinen constant [1], and $\rho_n$ and $\rho$ are the normal component density and total density respectively. $\alpha$, $\beta$, and $\mathbf{w}_0$ are taken to be constants in the calculations. The vortex interaction algorithm is made non-local by the reconnection ansatz which states that whenever two vortices cross they reconnect. The reconnection is unique since the vortex has a direction associated to it by the direction of the vorticity. The numerical calculations in [7] assume triply periodic boundary conditions and follow the evolution of vortex configurations which initially consist of a few vortex rings. It is found that the line length present in the periodic cube eventually reaches an equilibrium value which fluctuates about some constant value. Vinen argues heuristically [5] that

$$L = l_0(|\mathbf{w}_0|/\beta)^2, \qquad (2)$$

and Schwarz obtains numerical results which agree with Eq. (2). We obtain results which agree with Eq. (2) only when we underresolve the numerical computation.



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When the numerical computations are properly resolved we do not obtain the scaling given by Eq. (2) [13, 14].

For the numerical calculations Schwarz makes Eq. (1) dimensionless by arbitrarily choosing a length scale $l$ and a time scale $t_0 = l^2/\beta$; with this choice of units, Eq. (1) becomes [7, 8]

$$\frac{\partial \mathbf{r}}{\partial t} = \mathbf{r}' \times \mathbf{r}'' + \alpha \frac{l}{\beta} \mathbf{r}' \times \mathbf{w}_0 + \alpha \mathbf{r}''. \tag{3}$$

$\beta = 1.0 \times 10^{-3}\ \mathrm{cm}^2/\mathrm{s}$ in the numerical simulations; a typical value of $\alpha$ is 0.1 and a typical value for $l\,|\mathbf{w}_0|/\beta$ is 40.

There is an inherent length scale present in Eq. (1) equal to $\beta/|\mathbf{w}_0|$ and an inherent time scale equal to $\beta/(|\mathbf{w}_0|^2)$; we define the dimensionless position $\boldsymbol{\chi} = \mathbf{r}\,|\mathbf{w}_0|/\beta$ and the dimensionless time $\tau = t\,|\mathbf{w}_0|^2/\beta$ and obtain from Eq. (1) that

$$\frac{\partial \boldsymbol{\chi}}{\partial \tau} = \boldsymbol{\chi}' \times \boldsymbol{\chi}'' + \alpha \boldsymbol{\chi}' \times \hat{\mathbf{w}}_0 + \alpha \boldsymbol{\chi}'', \tag{4}$$

where $\hat{\mathbf{w}}_0$ is a unit vector in the $\mathbf{w}_0$ direction. Although Schwarz was apparently aware of the length scale $\beta/|\mathbf{w}_0|$ [6] in Eq. (1) he chose the scaling given in Eq. (3) for his numerical simulations.

A detailed analysis of Eq. (4) shows that in these dimensionless units in order to accurately solve Eq. (4) the numerical mesh spacing $\Delta\xi$ must be chosen so that

$$\Delta\xi \ll 1 \tag{5}$$

for both the numerical scheme used by us [14] and for the numerical scheme used by Schwarz [9]. If the condition in Eq. (5) is written in physical units we find that the spacing between points on the vortex $\Delta r$ must satisfy the condition:

$$\Delta r \equiv \frac{\beta\, \Delta\xi}{|\mathbf{w}_0|} \ll \frac{\beta}{|\mathbf{w}_0|}. \tag{6}$$

Thus it is not sufficient to simply keep the mesh spacing "small compared to the smallest radius of curvature of the line" as Schwarz does [15]. Spurious loops will form even though the reconnection is artificially smoothed and the reconnection appears to be properly resolved in terms of its maximum curvature.

Equation (6) indicates the reason one does not want to scale Eq. (1) to obtain Eq. (3) as is done in [7, 8]. If $\Delta r$ is held fixed and $|\mathbf{w}_0|$ is varied as is done in [7, 8] the actual resolution of the calculation is changing.

Although Schwarz readily admits that vortex loop generation is responsible for maintaining the vortex tangle [7], he also assures us that it is trivial to establish that his simulations never produce spurious vortex loops [16]. Schwarz claims to be able to make the reconnection in such a way that even though the calculation is underresolved the spurious loops will not form [16]; this is an interesting claim which is not trivial. An implementation of an algorithm which would prevent the formation of spurious loops without properly resolving the vortex would require a detailed knowledge of the long-time behaviour of the general solution of Eq. (4) for every possible angle between two vortices when they cross and for every possible orientation of the vortices relative to $\hat{\mathbf{w}}_0$; we are not aware that any such information is available. The only way one is able to determine which evolution is correct is to properly resolve the calculation and thereby calculate the solution to the evolution equation. Our calculations show that when the numerical computation is underresolved the long-time asymptotic state is unstably dependent on the manner in which the reconnection is made even if it is artificially smoothed as in [16].

Schwarz actually observed this spurious loop generation in some of his calculations [8] in which he observed that he could change his results by a factor of 2 to 4, depending on how he made the reconnections in his simulations. Schwarz observed that when he "artificially inhibited the surface reconnections" he could reduce the line length density.

There is one final claim made by Schwarz which is irrelevant to the question at hand: that is the question of the randomizing procedure, which is now claimed to be a key ingredient of the procedure but had not been divulged for five years [16]. This procedure is not necessary for small values of the parameter $\alpha$ as we show in [14]. Since we *are* able to reproduce the incorrect numerical results which agree with Eq. (2) and which agree with the results Schwarz obtained, it is irrelevant that our calculations do not include the randomizing step. The fact that we do not add a randomizing step has absolutely no relevance to the question at hand.

The only relevant question is whether one can numerically solve Eq. (4) when the equation is underresolved. The unsurprising answer to this question is that one cannot accurately solve Eq. (4) when it is underresolved.

This controversy can be resolved simply if the actual initial conditions used for the simulations presented in [7, 8] are made available. Equation (1) along with the reconnection ansatz specifies a unique evolution for a system of vortices and it is a simple matter to check the solutions by properly resolving the calculations using either of the two available algorithms [9, 13]. We have working versions of both algorithms which we are happy to make available to anyone who wishes to calculate the solutions of the evolution of the vortex systems.

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